ON FINITE SETS OF POINTS IN P^3

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ABSTRACT

Given a finite set of points Γ_0 which span a projective space P^3, we show here that a plane spanned by points of Γ_0 can be a neighbour of at most eight points of Γ_0, these being the vertices of a projective cube; the common neighbour plane is then elementary with the three only points of Γ_0 in it being diagonal points of the cube. This extends to P^3 some results of L. M. Kelly and W. O. J. Moser in the plane P^2.

1. Introduction

Let Γ_0 denote a finite non-planar set of points in a three-dimensional ordered projective space P^3. A plane α spanned by points of Γ_0 was called-by Motzkin [4]—"ordinary" if all but one of the points of $\alpha \cap \Gamma_0$ are collinear; if α contains exactly three points of Γ_0, then it is called "elementary". Motzkin showed, in [4], that every set Γ_0 determines at least one ordinary plane. This result is an extension to P^3 of Sylvester's theorem concerning finite planar non-linear sets of points. In [2], Hansen extended the result of Motzkin to arbitrary finite dimension d of the space.

If P is a planar non-linear set of n points (n finite) call a line spanned by points of P "ordinary" if it contains exactly two points of P. Denoting by m the number of ordinary lines determined by P, Kelly and Moser showed in [3] that $m \geqq 3n/7$. To obtain this result, they defined and investigated the "residence" and the "neighbours" of a point of P. One of their results is the following: a line s can be a neighbour of at most four points (of P), and if s is a neighbour of exactly four points, then s joins two diagonal points of the complete quadrangle determined by the four points, s is ordinary, the only two points of P on s being these diagonal points (corollary 3.4 and following remark in [3]).

The purpose of the present paper is to extend to P^3 some of the results of Kelly

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and Moser. The main result here obtained is the following extension of the above-mentioned theorem concerning a complete quadrangle:

FUNDAMENTAL THEOREM. A plane α can be the neighbour of at most *eight* distinct points of Γ_0. If α is the neighbour of exactly eight points of Γ_0, these points are the vertices of a projective cube, the plane α is elementary and the points of $\Gamma_0 \cap \alpha$ are three (out of four) diagonal points of the cube.

A "projective cube" is defined below; it is, in a sense, an extension to P^3 of a complete quadrangle (projective square).

2. Definitions, notation and terminology

In what follows Γ_0 is a finite non-planar set of points in an ordered projective space P^3, Γ_1 denotes the set of lines spanned by points of Γ_0, and Γ_2 the set of planes spanned by points of Γ_0.

Points are denoted by capital Latin letters $A, B, C, \cdots$, lines by small Latin letters $p, q, r, \cdots$, and planes by small Greek letters $\alpha, \beta, \gamma, \cdots$. For elements specifically belonging to Γ_0, Γ_1 or Γ_2, the superscript ($^\circ$) will be added; e.g., A° B° are points of $\Gamma_0, q^\circ \in \Gamma_1, \alpha^\circ \in \Gamma_2$, while B, r, γ are not necessarily elements of the corresponding set Γ_i. Subscripts are used to distinguish particular points, lines or planes.

If the plane α° is spanned by A° and q°, one writes: $\alpha^\circ = A^\circ q^\circ$; and if α° is spanned by the three noncollinear points $A^\circ, B^\circ, C^\circ$ (or A_i°, $i = 1, 2, 3$), one writes: $\alpha^\circ = A^\circ B^\circ C^\circ$ or $\alpha^\circ = A_1^\circ A_2^\circ A_3^\circ$.

Following the terminology in [1], if α° is ordinary, but not elementary, the points of $\Gamma_0 \cap \alpha^\circ$ lie all but one on a line called the "follower" line of α°, while the unique point of $\Gamma_0 \cap \alpha^\circ$ not on this line is the "leader" of α°.

A finite non-empty set of planes determines a partition of space into convex regions bounded by these planes; each such closed region is a three-dimensional cell δ_3. Let $\Gamma_2(A^\circ)$ be the subset of planes of Γ_2 not incident to A°; then A° is interior to one (and only one) cell $\delta_3(A^\circ)$ determined by $\Gamma_2(A^\circ)$, this cell is the "residence" of A°, while each of the planes, bounding it, is called a "neighbour" of A°. This terminology is an extension to P^3 of the one introduced, for P^2, in [3]. Definitions and properties of convex regions, boundary points, lines or planes are those given in [5].

The notion of "separation" of four distinct collinear points is fundamental in the proofs of Sylvester's and Motzkin's theorems and of other results derived from these latter. In the present paper, the plane and the space duals of this concept are extensively used. If A, B separate C, D we write $AB \parallel CD$, with similar notations for lines and planes. For more details on this concept see, e.g., [5].

The "*projective cube*": consider a cube in a Euclidean space E^3 (or a parallelepiped in an affine space A^3) with its eight vertices $A_i (i = 1, 2, \cdots, 8)$, the three points at infinity C_1, C_2, C_3 on its edges, and its center C_4. The configuration formed by these twelve points, together with its 16 edges and 12 face planes (or any projectively equivalent configuration) is called here a "projective cube". The A_i's are the vertices of the cube ($1 \leqq i \leqq 8$), the C_j's are its "diagonal points" ($j = 1, 2, 3, 4$). Each C_j is incident to four lines joining pairs of vertices, these are the "edges" of the projective cube. The eight vertices can be partitioned—in six different ways—in two sets of four coplanar points which are the vertices of two complete quadrangles lying in two different planes (said to be "opposite faces" of the cube) in such a way that these quadrangles have two of the C_j's as common diagonal points, and are in perspective to each other from each of the two remaining C_j's; the projective cube has thus six pairs of opposite face planes, and in all 12 face planes.

A complete quadrangle is projectively equivalent to a square (or parallelogram) two of its diagonal points correspond to the points at infinity on the edges of the square, while the third diagonal point corresponds to its center. With this in mind, a "projective cube" is an extension to P^3 of a complete quadrangle.

3. Common neighbour planes

In this section, we consider a plane α° which is a neighbour of k points $A_i^\circ \in \Gamma_0$, $(i = 1, 2, \cdots, k)$ and we investigate the possible configurations formed by these k points and by the set $\alpha^\circ \cap \Gamma_0$, for different values of k. The following terminology will be used: let $A_{ij} = A_i^\circ A_j^\circ \cap \alpha^\circ$ with $i \neq j$, $i, j = 1, 2, \cdots, k$, then A_{ij} will be called an "int.-point" (intersection point) in α° w.r. to (with respect to) the set $\{A_i^\circ\}$. If $B^\circ \in \Gamma_0 \cap \alpha^\circ$, and B° is not an int.-point, we call it an "outsider" w.r. to $\{A_i^\circ\}$.

LEMMA 1. *Three distinct collinear points $A_i^\circ (i = 1, 2, 3)$ cannot have a common neighbour plane α°.*

PROOF. Let α° be a plane of Γ_2 not incident to any of the A_i°'s, and let $B = \alpha^\circ \cap A_1^\circ A_2^\circ$; we may assume the points labelled so that:

$$A_1^\circ B \parallel A_2^\circ A_3^\circ \tag{1}$$

In α°, there must exist a line q° not through B; let $\beta_i^\circ = q^\circ A_i^\circ (i = 1, 2, 3)$. Then the four planes α° and β_i° have q° as common axis and from (1) follows:

$$\beta_1^\circ \alpha^\circ \| \beta_2^\circ \beta_3^\circ \tag{2}$$

which implies that α° is not a neighbour of A_1°.

LEMMA 2. *If the plane α° is a neighbour of three non collinear points A_i° ($i = 1, 2, 3$), then every line $q^\circ \in \Gamma_1$ in α° must be incident to an int.-point A_{ij}.*

PROOF. Let $\beta^\circ = A_1^\circ A_2^\circ A_3^\circ$ (the plane spanned by the A_i°'s) and $m = \alpha^\circ \cap \beta^\circ$, then the three int.-points A_{12}, A_{13}, A_{23} are distinct and lie on m. If some $q^\circ \in \alpha^\circ \cap \Gamma_1$ meets m in a point Q different from the A_{ij}'s, then the four planes α° and $\gamma_i^\circ = q^\circ A_i^\circ$ ($i = 1, 2, 3$) are distinct and have q° as common axis. We may assume the points labelled so that:

$$\alpha^\circ \gamma_1^\circ \| \gamma_2^\circ \gamma_3^\circ \tag{3}$$

As A_1° lies in γ_1°, relation (3) implies that this point A_1° is separated from any point in α° (not on q°) by γ_2° and γ_3°, so α° cannot be a neighbour of A_1°, which is a contradiction. Hence one must have $Q = A_{ij}$ for some i and j.

With the same notation as in Lemma 2, the following theorem is a direct result of this Lemma and the proof will be omitted:

THEOREM 1. *If α° is a neighbour of three non-collinear points $A_i^\circ (i = 1, 2, 3)$, then the points of $\Gamma_0 \cap \alpha^\circ$ form one of two possible configurations:*

—type "L": consists of a set of points collinear with one of the A_{ij}'s (including possibly this latter) and one or both of the two other A_{ij}'s;

—type "T": consists of three points $B_i^\circ (i = 1, 2, 3)$ vertices of a triangle such that each $B_i^\circ B_j^\circ$ is incident with the corresponding A_{ij}, also some or all of the A_{ij}'s can be elements of Γ_0.

REMARK 1. In type "T", the two triangles $A_1^\circ A_2^\circ A_3^\circ$ and $B_1^\circ B_2^\circ B_3^\circ$ are in perspective from the line $m = \alpha^\circ \cap \beta^\circ$ (β° being the plane spanned by the A_i°'s). Then, by the converse of Desargues' theorem, the two triangles are in perspective from some point M such that $A_i^\circ B_i^\circ$ are collinear with M for each $i = 1, 2, 3$.

Now we consider four points A_i° ($i = 1, 2, 3, 4$) which are the vertices of a tetrahedron in space; the four face planes of this tetrahedron intersect a plane α°—not incident to any of the A_i's—in four lines m_i where:

$m_i = \alpha^\circ \cap A_j^\circ A_h^\circ A_k^\circ$ and i, j, h, k is a permutation of the integers $1, 2, 3, 4$. The m_i's in α° are the sides of a complete quadrilateral, the six vertices of which are the int.-points A_{jk} (clearly if A_{jk} lies on m_i, then i, j, k are different from one another).

Two points A_{ij}, A_{hk} are called "opposite" int.-points iff they are opposite vertices of the complete quadrilateral; clearly opposite vertices correspond to opposite edges $A_i^\circ A_j^\circ, A_h^\circ A_k^\circ$ of the tetrahedron and conversely.

$A_{12}A_{34}, A_{13}A_{24}, A_{14}A_{23}$ are the sides of the diagonal triangle of the quadrilateral, and let B_1, B_2, B_3 be the vertices of this triangle; with this notation, one obtains the following:

THEOREM 2. *If α° is a neighbour of four (not coplanar) points $A_i^\circ (i = 1,2,3,4)$ then the points of $\Gamma_0 \cap \alpha^\circ$ are among the nine points $A_{ij}, B_k (i,j = 1,2,3,4\ i \neq j;$ $k = 1,2,3)$ and three configurations only are possible:*

—type 1: consists only of int.-points A_{ij}, possibly all of them;

—type 2: consists of one B_k and up to four A_{ij}'s (those four which lie on the two sides of the diagonal triangle meeting at B_k):

—type 3: consists of the three B_k's only, *and α° is an elementary plane.*

PROOF. Applying theorem 1 to each triple of the A_i°'s, one sees easily that the six A_{ij}'s form a configuration of type "T" (or of type "L" if one takes less than the six points), thus type 1 is certainly possible. Suppose next that $\alpha^\circ \cap \Gamma_0$ consists of at least one int.-point, say A_{12} (denoted then A_{12}°), and at least one outsider, call it X°. By Lemma 2 X° cannot lie on any m_i and A_{12}° does not lie neither on m_1 nor on m_2; then by applying Lemma 2 to each of the triples $A_2^\circ A_3^\circ A_4^\circ$ and $A_1^\circ A_3^\circ A_4^\circ$, it follows that $X^\circ A_{12}^\circ$ must be incident to A_{34} the int.-point opposite to A_{12}°. Thus, every outsider other than X° should also lie on this line $A_{12}^\circ A_{34}$, which is a side of the diagonal triangle. However, there exist a point of $\Gamma_0 \cap \alpha^\circ$ not collinear with $A_{12}^\circ X^\circ$; such point is then another int.-point; assume it to be A_{13} (hence denoted A_{13}°). The int.-point opposite to A_{13}° is A_{24}, and repeating the above argument, X° and every outsider should lie also on $A_{13}^\circ A_{24}$ (a second side of the diagonal triangle). Hence there can be only one outsider: the point of intersection of $A_{12}^\circ A_{34}$ and $A_{13}^\circ A_{24}$, i.e., one of the B_k's; the other possible points are evidently A_{34} and A_{24} and only these two. This is type 2.

Finally, suppose α° contains only outsiders; at least three non-collinear ones are required. By Lemma 2 applied to each of the triples $A_i^\circ A_j^\circ A_k^\circ$, these outsiders cannot lie on any of the m_h's. Then, by an argument similar to the one above, each pair of outsiders must be collinear with two opposite int.-points, i.e. lie on a side of the diagonal triangle; as we have three pairs not on one line, these outsiders must coincide with the B_i's and there can be no other points: this is type 3. Of course α° is elementary in this case.

All possible combinations of int.-points and outsiders have been investigated, and the proof of the theorem is complete.

REMARK 2. As noted in Remark 1, the triangle $B_1B_2B_3$ is in perspective with each of the triangles $A_1^\circ A_2^\circ A_4^\circ$, $A_1^\circ A_3^\circ A_4^\circ$, $A_2^\circ A_3^\circ A_4^\circ$ and $A_1^\circ A_2^\circ A_3^\circ$. If A_5, A_6, A_7, A_8 are the respective centres of perspectivity, it is easily shown that the eight points: $A_i^\circ (i = 1,2,3,4)$ and $A_j (j = 5,6,7,8)$, are the vertices of a projective cube, the points $B_k (k = 1,2,3)$ being three (out of four) of the diagonal points of the cube. The four A_i°'s and the plane α° completely determine the cube.

THEOREM 3. *If a plane α° is a neighbour of five points A_i° $(i = 1,2,\cdots,5)$ no four of which being coplanar, then α° is an elementary plane; the three points of $\Gamma_0 \cap \alpha^\circ$ are int.-points w.r. to the five A_i°'s. Moreover, the indices can be chosen so that these points will be $A_{51}^\circ, A_{52}^\circ, A_{53}^\circ$ so that α° is of type 1 w.r. to $A_5^\circ A_1^\circ A_2^\circ A_3^\circ$, of type 3 w.r. to $A_1^\circ A_2^\circ A_3^\circ A_4^\circ$ and of the type 2 w.r. to each of the three other possible tetrahedra. Except for a permutation of subscripts, this is the only possible configuration.*

PROOF. The five points A_i° are vertices of five tetrahedra. By Theorem 2, α° should be of one of the types 1, 2 or 3 w.r. to each tetrahedron. The proof of the theorem will be given in a series of steps or lemmas.

LEMMA 3. *α° can be of type 3 w.r. to at most one of the tetrahedra.*

PROOF. If α° is of type 3 w.r. to two tetrahedra—say, e.g., $A_1^\circ A_2^\circ A_3^\circ A_4^\circ$ and $A_1^\circ A_2^\circ A_3^\circ A_5^\circ$, then the two corresponding complete quadrilaterals in α° have the same diagonal triangle, and they also have a common edge $m_4 = m_5 = \alpha^\circ \cap A_1^\circ A_2^\circ A_3^\circ$ as the plane $A_1^\circ A_2^\circ A_3^\circ$ is a face of both tetrahedra. However a well-known and easily proved exercise in elementary planar projective geometry states there is one and *only one* complete quadrilateral having a given triangle as diagonal triangle and a given line (not incident to any vertex of the triangle) as an edge. On the other hand, the two quadrilaterals cannot coincide ($A_4^\circ \neq A_5^\circ$), so we get a contradiction and the lemma is proved.

LEMMA 4. *All the points of $\alpha^\circ \cap \Gamma_0$ are int.-points A_{ij} w.r. to the five A_i°'s.*

PROOF. From Lemma 3 above, α° is of type 1 or type 2 w.r. to at least four of the tetrahedra, and therefore it contains at most one outsider point. Assume there is an outsider X°; then there must also be an int.-point $A_{ij} \in \Gamma_0$ for some values of i and j $(i \neq j)$. It follows that w.r. to the tetrahedron which has not

A_i° as vertex and w.r. to the tetrahedron which has not A_j° as vertex, A_{ij} would be a second outsider. This in turn implies α° is of type 3 w.r. to both tetrahedra, contradicting Lemma 3. Hence no X° exists.

LEMMA 5. *α° is of type* 3 *w.r. to* at least *one tetrahedron.*

PROOF. Suppose α° is not of type 3 w.r. to any of the five possible tetrahedra. By Lemma 4, $\alpha^\circ \cap \Gamma_0$ consists of at least three int.-points (non-collinear). We may assume A_{12} is one of them, then the indices 1 and 2 cannot appear in the other A_{ij}'s $\in \Gamma_0$, otherwise w.r. to the tetrahedron not having the corresponding A_1° or A_2° as vertex, α° would have (at least) two outsiders and be of type 3, contradicting the assumption. Then a second possible point would be A_{34} (or A_{35} or A_{45}). Again, by the same argument, a third point $A_{ij} \in \Gamma_0$ cannot have $1,2,3,4$ as indices; but only the index 5 remains and $i \neq j$; one gets an impossibility; and the lemma is proved.

Going back now to the proof of Theorem 3, it follows from Lemmas 3 and 5, that α° must be of type 3 w.r. to exactly one tetrahedron, and α° is then elementary. We may suppose the points labelled so that the point which is not a vertex of that particular tetrahedron (w.r. to which α° is of type 3) is A_5°. By Lemma 4, the three unique points of $\alpha^\circ \cap \Gamma_0$ are int.-points w.r. to the five A_i°'s but outsiders w.r. to the four $A_1^\circ, A_2^\circ, A_3^\circ, A_4^\circ$. This requires that these points be indexed $A_{5i}^\circ, A_{5j}^\circ, A_{5k}^\circ$, with i,j,k being three (distinct) out of the four integers $1,2,3,4$. Again one may assume the labelling so that these are the points $A_{51}^\circ, A_{52}^\circ, A_{53}^\circ$, which shows that α° is of type 1 w.r. to the tetrahedron $A_1^\circ A_2^\circ A_3^\circ A_5^\circ$. It is easily checked that α° is of type 2 w.r. to the three remaining tetrahedra; it is, of course, of type 3 w.r. to $A_1^\circ A_2^\circ A_3^\circ A_4^\circ$.

With the same notation as in Theorem 3 and using Remark 2, with the points B_i being the A_{5i}° $(i = 1,2,3)$ one proves without difficulty the following:

COROLLARY 1. *There is one and only one projective cube having the points A_i° $(i=1,2,3,4,5)$ as vertices and the points A_{5j}° $(j=1,2,3)$ as diagonal points. Furthermore, given any four of the five A_i°'s, and all three A_{5j}°'s, the fifth A_i°'s is uniquely determined by these seven points so that the plane α° spanned by the A_{5j}°'s be a neighbour of the five A_i°'s and the A_{5j}°'s be elements of Γ_0.*

COROLLARY 2. *Six or more points of Γ_0, no four of which are coplanar, cannot have a common neighbour plane.*

This is an obvious result of Theorem 3 and Corollary 1.

We investigate now the case of four points $A_i^\circ (i = 1,2,3,4)$ which are the vertices of a complete quadrangle in a plane β° and have a common neighbour plane α° with $m = \alpha^\circ \cap \beta^\circ$. We remark that if two opposite sides of the quadrangle say, e.g. $A_1^\circ A_2^\circ$ and $A_3^\circ A_4^\circ$ meet m (hence α°) in int.-points A_{12}, A_{34}, and $A_{12} \neq A_{34}$, it will follow from Lemma 2 that no line $q^\circ \in \Gamma_1 \cap \alpha^\circ$ is incident to A_{12} or A_{34}, as every triangle having three of the A_i°'s as vertices has one and only one of these opposite sides as an edge. Using this remark and again Lemma 2 the fol owing theorem is easily proved:

THEOREM 4. *If α° is a neighbour plane of four points A_i° $(i=1,2,3,4)$ which are vertices of a complete quadrangle in a plane β°, then two diagonal points of the quadrangle are incident to the line $m = \alpha^\circ \cap \beta^\circ$ (i.e. must be int.-points w.r. to the A_i°'s); also the plane α° is ordinary, one of these diagonal points on m being the leader point, while the follower line is incident to the second diagonal point on m, this latter point may be itself, a point of Γ_0.*

COROLLARY 3. *Five or more distinct coplanar points of Γ_0 cannot have a common neighbour plane.*

This is an immediate result of Theorem 4, as two complete quadrangles having three common vertices cannot have two common diagonal points.

THEOREM 5. *If α° is a neighbour of five points A_i° $(i = 1,2,3,4,5)$—where four of them are vertices of a complete quadrangle in a plane β°, while the fifth one, say A_5°, is not incident to β° then α° is ordinary with one of the diagonal points of the quadrangle as leader point, while the follower line is incident to a second diagonal point of the quadrangle; also this line contains at most three points of Γ_0: this second diagonal point itself and two int.-points A_{i5}. Hence $\alpha^\circ \cap \Gamma_0$ contains at most four points which are int.-points w.r. to the five A_i°'s.*

PROOF. Without loss of generality, one may assume that the points have been labelled so that A_i°, $(i = 1,2,3,4)$ are the vertices of the quadrangle, in the plane β°. By Corollary 3, A_5° does not lie in β°. By Theorem 4, α° being a neighbour of the four vertices of the quadrangle, it is ordinary with two diagonal points lying on $m = \alpha^\circ \cap \beta^\circ$, and again one may assume the points in β° labelled so that $A_{12} \equiv A_{34}$ and $A_{14} \equiv A_{23}$ are the two diagonal points on m, and so that $A_{12} \in \Gamma_0$ (hence denoted A_{12}°); then the follower line must be incident to A_{14}. It remains to show that on that line, there may be besides A_{14} itself at most two more points of Γ_0 and these are int.-points A_{i5}.

In what follows, $ijhk$ is any permutation of the integers 1234. Apply now

Theorem 2 to each of the four tetrahedra $A_5^\circ A_i^\circ A_j^\circ A_h^\circ$. The vertices of the corresponding complete quadrilateral in α° are the points A_{12}°, A_{14}, A_{i5}, A_{j5}, A_{h5} and either A_{13} or A_{24}, while the point A_{k5} is always a vertex of the diagonal triangle of the quadrilateral (this is easily verified for every permutation $ijhk$). Clearly, as $A_{12}^\circ \in \Gamma_0$, α° is of type 1 or type 2 w.r. to each tetrahedron and must be of type 2 w.r. to at least one of them, say $A_5^\circ A_i^\circ A_j^\circ A_h^\circ$; however the only possible vertex of the corresponding diagonal triangle which lies on that side of this triangle which is incident to A_{12}° is exactly A_{k5}.

Therefore, $\alpha^\circ \cap \Gamma_0$ consists of A_{12}°, A_{k5}, possibly A_{14} and the other A_{q5} which lies on the line $A_{14}\, A_{k5}$, i.e. together either the pair (A_{15}, A_{45}) or the pair (A_{25}, A_{35}) (both elements or only one of the pair). Then one verifies that with these four points (or only three of them) α° is of type 1 w.r. to each of the tetrahedra having both A_k° and A_q° as vertices and of type 2 w.r. to those tetrahedra having only one of them as vertex; and there are no other possibilities.

COROLLARY 4. *With the same conditions and notation as in Theorem 5 above, there exists a projective cube (not unique) having all five A_i° $(i = 1,2,3,4,5)$ as vertices and three (non-collinear) out of the four possible points of $\alpha^\circ \cap \Gamma_0$ as diagonal points.*

PROOF. α° must contain (at least) three non-collinear points of Γ_0, hence one of them must be either A_{12} or A_{14}, the other either A_{k5} or A_{q5} (with the notation of Theorem 5). Suppose the labelling is so that A_{12}° and A_{15}° are certainly elements of Γ_0, then either A_{14} or A_{45} (or both) are also elements of Γ_0. Then in the plane of $A_5^\circ A_1^\circ A_2^\circ$, the point $A_6 = A_{12}^\circ A_5^\circ \cap A_{15}^\circ A_2^\circ$ is the only possible fourth point, if a projective cube exists with the conditions as stated. In the plane of $A_5^\circ A_1^\circ A_4^\circ$, two possible cases arise: if A_{14} is taken as third diagonal point of the required cube, the only possible point in this plane is $A_8 = A_{14} A_5^\circ \cap A_{15}^\circ A_4^\circ$. Then, in the plane $A_3^\circ A_4^\circ A_8$ containing A_{12}° and A_{15}°, one obtains $A_7 = A_{12}^\circ A_8 \cap A_{15}^\circ A_3^\circ$. It follows that A_5°, A_6, A_7, A_8 are coplanar vertices of a complete quadrangle which has A_{12}° and A_{14} as diagonal points. It is easily seen that A_i° $(i=1,2,3,4,5)$ and A_j $(j = 6,7,8)$ are the eight vertices of a projective cube, with diagonal points A_{12}°, A_{15}°, A_{14}. If A_{45} is taken as third diagonal point, in the plane $A_5^\circ A_1^\circ A_4^\circ$, one gets the point $A_8' = A_{45} A_1^\circ \cap A_{15} A_4^\circ$. Then, in the plane $A_3^\circ A_4^\circ A_8'$ containing A_{12}° and A_{15}°, one obtains, as above: $A_7' = A_{12}^\circ A_8' \cap A_{15}^\circ A_3^\circ$. Again, it is easy to show that the eight points A_i° $(i = 1, 2, \cdots, 5)$ A_6, A_7' and A_8' are vertices of a cube with diagonal points A_{12}°, A_{15}°, A_{45}.

In the above discussion, A°_{12} and A°_{15} have been taken, in all cases, as diagonal points of the cube; other cubes can be obtained having $A^\circ_i (1 \leqq i \leqq 5)$ as vertices and other points A_{ij} as diagonal points.

THEOREM 6. *If a plane α° is a neighbour of six points of Γ_0, then it is an ordinary plane and contains at most four points of Γ_0. Furthermore, the six points are vertices of a projective cube having three of the points of $\alpha^\circ \cap \Gamma_0$ as diagonal points.*

PROOF. By Corollaries 2 and 3, four of the six points having α° as neighbour must lie in one plane β°, call them A°_i $(i = 1,2,3,4)$; the two others cannot be incident to β°. Call one of them A°_5 and apply Theorem 5 to the five A°_i's; this proves immediately the first part of the present theorem. As in the proof of Theorem 5, one may assume the points labelled so that the two diagonal points of the quadrangle A°_i $(i=1,2,3,4)$ are A°_{12} and A_{14}, with A°_{12} and A°_{15} elements of Γ_0, and possibly A_{45} and A_{14} (one of them at least) also. Let A°_q be the sixth point for which α° is a neighbour and apply again Theorem 5 to the set of five points $A^\circ_1, A^\circ_2, A^\circ_3, A^\circ_4, A^\circ_q$. Any point of $\alpha^\circ \cap \Gamma_0$ must be an int.-point of this set, hence one must have $A^\circ_{15} \equiv A_{jq}$ for some value of j taken among the integers 2, 3, 4 ($j = 1$ would imply that $A^\circ_1, A^\circ_5, A^\circ_q$ are collinear, contradicting Lemma 1). Thus the points $A^\circ_1, A^\circ_5, A^\circ_j, A^\circ_q$ are coplanar and one can apply Theorem 4 to this set. Together with the results applying to the set A°_i $(i = 1, 2, \cdots, 5)$ stated above, three cases are possible:

(i) $j = 2$; the plane $A^\circ_1 A^\circ_5 A^\circ_2$ contains already the points A°_{12} and A°_{15}, as elements of Γ_0, then A°_q must coincide with the point A_6 of Corollary 4. Both A_{45} and A_{14} are possible elements of Γ_0.

(ii) $j = 3$; the plane $A^\circ_1 A^\circ_5 A^\circ_3$ contains $A^\circ_{15} \in \Gamma_0$; thus this point is one of the diagonal points of $A^\circ_1 A^\circ_5 A^\circ_3 A^\circ_q$, i.e. A°_q lies on $A^\circ_3 A^\circ_{15}$. The other diagonal point (in α°) must be either A_{13} or A_{35}. If it is A_{13}, then $A^\circ_q = A_{13} A^\circ_5 \cap A^\circ_{15} A^\circ_3$. It can be verified that this is point A_7 of Corollary 4, and in this case, as $A_{13} \notin \Gamma_0$, the only possible third point of $\Gamma_0 \cap \alpha^\circ$ is A_{14}, ($A_{14} A^\circ_{12}$ is incident to A_{13}, as required by Theorem 4); α° is elementary. If the required diagonal point in α° is A_{35}, then: $A^\circ_q = A_{35} A^\circ_1 \cap A^\circ_{15} A^\circ_3$ and $A^\circ_q \equiv A'_7$ of Corollary 4; α° is again elementary with A_{45} as third point of $\alpha^\circ \cap \Gamma_0$.

(iii) $j = 4$; A°_q must lie in the plane $A^\circ_1 A^\circ_4 A^\circ_5$ which contains A°_{15}, A_{14} and A_{45}. Then, according to which of A_{14} or A_{45} is taken as second diagonal point of the corresponding quadrangle, and by arguments similar to those in (ii), one obtains respectively $A_q^\circ \equiv A_8$ or A_8' (of Corollary 4). In the first case, $\alpha^\circ \cap \Gamma_0$ consists of $A_{12}^\circ, A_{15}^\circ, A_{14}$ and also possibly A_{25}; in the second case, $\alpha^\circ \cap \Gamma_0 = \{A_{12}^\circ, A_{15}^\circ, A_{45}\}$, α° is elementary.

The second part of the theorem follows from the above results and Corollary 4.

THEOREM 7. *If a plane α° is a neighbour of seven distinct points of Γ_0, then these points are vertices of a projective cube, α° is elementary, and the three points of $\alpha^\circ \cap \Gamma_0$ are diagonal points of the cube.*

PROOF. As in the proof of Theorem 6, four of the points must lie in a plane β° call them A_i°, $i = 1,2,3,4$; the three others are not in β°, call one of them A_5°, and w.r. to those five points A_i° $(1 \leqq i \leqq 5)$, with the same notation as in Theorems 5 and 6, suppose A_{12}°, $A_{15}^\circ \in \Gamma_0 \cap \alpha^\circ$. Then, by similar arguments as above, and by Theorem 6, it follows that for the two other points having α° as neighbour, the only possible cases are:

(i) The two points are the points A_6 and A_7, or A_6 and A_8, or A_7 and A_8 (with the notation of Corollary 4); in each of these cases, α° is elementary, with A_{14} as third point of Γ_0 in α°.

(ii) The two points are the points A_6 and A_7', or A_6 and A_8', or A_7' and A_8', again α° is elementary with $A_{45} \in \Gamma_0$.

Then the theorem follows from these facts and Corollary 4.

The Fundamental Theorem, stated in the Introduction, now follows in a natural way: With the same notation as in Corollary 4 and Theorems 5, 6, 7, if α° is a neighbour of eight points, then clearly, these are:

— either A_i° $(i = 1,2,\cdots,5)$, A_6, A_7, A_8 and $\alpha_0 \cap \Gamma_0 = \{A_{12}^\circ, A_{15}^\circ, A_{14}^\circ\}$; or A_i° $(i = 1,2,\cdots,5)$, A_6, A_7', A_8' with $\alpha_0 \cap \Gamma_0 = \{A_{12}^\circ, A_{15}^\circ, A_{45}^\circ\}$, and obviously, no more than eight points can have a common neighbour plane.

4. Conclusion

The Fundamental Theorem proved here for P^3 is an extension of corresponding results of Kelly and Moser for the plane case, as stated in the Introduction (see [3]) and this suggests the following conjecture:

Given, in a d-dimensional ordered projective space P^d, a set Γ_0 of n points (n finite) not lying in one hyperplane, with the corresponding definitions of the residence of a point, a neighbour hyperplane, and a projective d-dimensional cube with 2^d vertices and $d+1$ diagonal points, one may conjecture that a hyperplane can be a neighbour of at most 2^d points of Γ_0, these points being the vertices of a projective cube, the hyperplane being then elementary, with exactly d points of Γ_0 incident to it, these d points being diagonal points of the cube.

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